Optimal bank capital with costly recapitalization*

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Abstract

We study optimal bank capital choice as a dynamic tradeoff between the opportunity cost of equity, the loss of franchise value following a regulatory minimum capital violation, and the cost of recapitalization. Our model indicates that a recapitalization option may be valuable despite substantial delays and costs of capital issuance, and that a significant fraction of the value of low capitalized banks may be attributable to the option to recapitalize. We calibrate the model to bank accounting return data and evaluate the model's ability to explain observed bank capital ratios. We find that the model has the potential to replicate a significant amount of the cross-sectional variation in bank capital ratios by relating these to differences in return volatility. Differences in the level of capital market imperfections across banks constitute a secondary explanation. Our analysis points to the need for improved forward looking estimates of bank return volatility.

Keywords: bank capital, dividends, capital issues, fixed cost, delay

JEL classification: G32, G35

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1 Introduction

A general risk management lesson from models with frictions is that, in the absence of explicit risk management tools such as financial derivatives, firms may choose to hold buffer stocks of liquid assets and capital as hedges against liquidity and earnings risks. The argument for the buffer stock role of liquid assets has been theoretically presented and empirically verified e.g. by Kim et al. (1998) and Opler et al. (1999), who find that firm liquid asset holdings are positively related to cash flow risks. The buffer stock role of equity capital, on the other hand, is supported by many empirical studies on capital structure (e.g. Harris and Raviv, 1990, Booth et al., 2001, and Titman and Wessels, 1988), who find that firm leverage is negatively related to earnings volatility. In other words, we observe risk management considerations to influence both corporate investment decisions and corporate financing decisions[1].

The capital structure decision in banks is in its very essence a risk management decision. A bank practitioner views bank capital[2] not primarily as a form of financing, but as a buffer against asset risks which needs to be managed so that the bank can satisfy its minimum capital requirement even under relatively adverse future scenarios. It is implicit in this view that the violation of the minimum capital requirement is costly for the bank, and that the bank faces costs and/or constraints associated with portfolio adjustment and recapitalization. Subject to these conditions the role of buffer capital as a hedging mechanism against minimum capital violation is well founded.

Much of academic banking theory has been built on a quite different view on bank capital. A sizable literature has studied conditions or regulatory set-ups which could eliminate bank owners' asset substitution moral hazard problem[3]. This literature concentrates on incentives in asset risk choice while taking bank capital as exogenous and abstracting from dividend and recapitalization choices. Therefore the

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[1] By now there is a large literature on the interactions between financing, investment, and risk management in the presence of capital market frictions (e.g. Acharya et al., 2000, Froot et al., 1993, Froot and Stein, 1998, Leland, 1998, Mello and Parsons, 2000).

[2] Consistent with common bank parlance, our use of the term bank capital refers to banks' book equity. This is the relevant measure of capital in an analysis of bank capital adequacy since minimum capital requirements under the Basel Accord apply to book equity.

[3] This literature includes Kahane (1977), Merton (1977), Koehn and Santomero (1980), Green (1984), Marcus (1984), Crouhy and Galai (1991), Rochet (1992), Fries et al. (1997) and Bhattacharya et al. (2002).

literature is ill suited for explaining banks' actual capital choices[4]. However, the two points of view on bank capital are complementary. The literature which builds on risk shifting incentives provides a rationale for banking regulation in general and for minimum capital regulation in particular. A bank manager thinking on bank capitalization takes the minimum capital constraint (the prevailing form of banking regulation) as well as the consequences of minimum capital violation as additional constraints to her choice. In a minimum capital regime such as the current Basel regime, a bank's capital choice is really a choice of the capital buffer to be held in excess of the minimum requirement. Asset risks, recapitalization constraints and the penalty from regulatory capital violation are the key determinants of this choice.

In this paper we adopt the bank manager's point of view on bank capital and test whether a simple optimizing model built on this view is capable of explaining the observed patterns of bank capital holdings. Our model is one of a value maximizing bank and prescribes an illiquid bank portfolio, imperfections in capital raising transactions, and loss of franchise value associated with the violation of the minimum capital requirement. Our model does not display any risk-shifting opportunities, since the bank portfolio is assumed illiquid. The model predicts strictly positive levels of buffer capital. We estimate the model parameters from bank level accounting returns and compare the implied bank capital ratios with actual bank capital ratios. We are not aware of a similar calibration exercise performed in the existing banking literature.

The idea to model banks' capitalization decision based on the buffer stock role of bank capital is not new. Our model builds on the basic continuous time model of a capital constrained firm presented in Milne and Robertson (1996). Hojgaard and Taksar (1999) have extended the basic model into an insurance company setting by assuming that risk reduction at a proportional cost is available, which is interpreted as cheap reinsurance. Milne and Whalley (2001) have extended the model to allow for a recapitalization option. Peura (2002) has analyzed the effects of an equity issuance option which is subject to a proportional cost and an upper bound on the rate at which external capital can be raised. Our main modeling innovation is to allow for a

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[4] This point has been emphasized by Milne (2002).

delay in the recapitalization of the bank[5]. We suggest that the delay is a natural stylized assumption which proxies for the fact that capital raising transactions in reality require heavy preparatory work. It turns out that the delay has a significant effect on the qualitative nature of the resulting capital raising policy. In the presence of a delay, bank liquidation probability is also strictly positive, unlike in a model where recapitalization can be implemented instantaneously.

The variants of the Milne and Robertson (1996) model are formulated in continuous time, and rely on stochastic control techniques to obtain optimal policies. The determination of bank capital from a trade-off perspective has also been studied in discrete time by Froot and Stein (1998), Estrella (2001) and Furfine (2000). In the context of a model with costly external capital, Froot and Stein (1998) demonstrate that a bank investing in illiquid products may adjust its capital structure in order to accommodate the illiquid risks it chooses to bear. Estrella (2001) uses a variant of the classical inventory or cash management models to study cyclicality of bank capital (see e.g. Karlin and Taylor, 1981, pp. 211-212 for the classical cash inventory model). In his model, the objective is to minimize the combined costs from over- and undercapitalization, as well as from adjustment of capital. Furfine (2000) presents a model of a value maximizing bank and calibrates the model to panel data in an attempt to explain banks' portfolio shifts following the introduction of the Basel Accord in 1989.

We estimate the model parameters based on accounting returns and capital for a sample of US commercial banks' in S&P's Compustat database. The data indicates that actual capital buffers are sizable: average total capital ratio over the 1994-2002 period in the sample is over 13%. The minimum requirement imposed by the Basel Accord is 8%, implying that the average capital buffer is in excess of 5% of risk-weighted assets, or over 50% of the minimum requirement. Our model, implemented with the means and volatilities of bank returns, yields an average capital buffer of 2.5% for the sample banks. The model capital ratio is highly sensitive to the level of return volatility, however, and much of the shortfall between model and actual capital ratios is attributable to banks with below average return volatilities. There is

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[5] The effects of delays on irreversible investment decisions have been studied in Bar-Ilan and Strange (1996) and Alvarez and Keppo (2001), and by Subramanian and Jarrow (2001) in connection with optimal liquidation.

reason to suspect that these estimates suffer from a peso problem due to the insufficient length of the data history as well as skewness and autocorrelation present in the distribution of bank returns. Smoothing of accounting returns, which arguably takes place through banks' discretionary credit loss provisions, may constitute a partial explanation for the low observed volatilities. We find that forward looking volatility estimates could improve model fit, both in terms of average levels and cross-sectionally. Our model can replicate observed capital ratios exactly if the volatility is made an implied parameter, analogously to how the Black-Scholes model is used in practice. The implied volatility for smaller banks appears higher than for large banks, even after acknowledging the likely differences in capital market imperfections.

The rest of the paper is organized as follows. Section 2 presents our model of capital control and shows that the non-homogenous problem of capital control can be transformed into a homogenous problem of capital *ratio* control. The solution is characterized in terms of a set of variational inequalities. Optimal policies and the value function, also in the limiting cases of the model, are derived in section 3. Section 4 illustrates optimal policies through numerical examples. Model parameters are calibrated and the comparison against actual capital ratios is presented in section 5. Section 6 concludes.

2 The model

2.1 A model of bank capital

We imagine a bank whose portfolio size, measured by its regulatory risk weighted assets[6] R, grows at a constant positive rate r, so that

$$R_t = R_0 e^{rt} \tag{1}$$

for some initial positive R_0. The growth rate r is assumed to coincide with the riskless rate. We assume that the bank's relative profitability remains unchanged over time,

[6] Risk weighted assets, under the current Basel Accord from 1988, are calculated as a weighted sum of a bank's nominal exposures, where the weights depends on product type and counterparty sector. For large banks, risk weighted assets are typically between 65 and 70 percent of total assets.

so that the scale of the bank's profits also grows at the rate r. Cumulative bank profit Y_t therefore satisfies

$$dY_t = \mu R_t dt + \sigma R_t dW_t ,$$

where W_t is a standard Wiener process[7]. The parameter μ is the expected return on (risk weighted) assets and σ is the asset return volatility. These are both assumed to be positive constants. This implies that expected instantaneous bank profit is proportional to bank portfolio size (measured by risk weighted assets), and that instantaneous profit is stochastic and non-predictable.

Owners control bank capital through dividend payments and issues of new capital. Dividends payments can be implemented instantaneously, but capital issuance is associated with a delay of length Δ and with a cost which is a fixed proportion K of the size of the bank (as measured by R). Formally, a capital control policy $\hat{\pi}$ is a collection $\left(L^{\hat{\pi}}, \left\{t_i^{\hat{\pi}}, s_i^{\hat{\pi}}\right\}\right)$, where $L^{\hat{\pi}}$ is a non-decreasing process representing the cumulative amount of dividends paid under policy $\hat{\pi}$, $\left\{t_i^{\hat{\pi}}\right\}$ is an increasing sequence of order times of new capital issues, and $\left\{s_i^{\hat{\pi}}\right\}$ are the amounts of capital raised at each issue of capital. We denote by Π the class of admissible policies which satisfy: $L_t^{\hat{\pi}}$ is a non-decreasing right-continuous process adapted to F_t such that $L_{0-}^{\hat{\pi}} = 0$; each $t_i^{\hat{\pi}}$ is a stopping time of the filtration F_t; each $s_i^{\hat{\pi}}$ is measurable with respect to $F_{\left(t_i^{\hat{\pi}}+\Delta\right)-}$. Additionally, admissible controls satisfy

$$t_{i+1}^{\hat{\pi}} - t_i^{\hat{\pi}} \geq \Delta \text{ for all } i \geq 1 , \tag{2i}$$

$$dL_t^{\hat{\pi}} = 0 \text{ for all } t \in \left(t_i^{\hat{\pi}}, t_i^{\hat{\pi}} + \Delta\right], \quad i \geq 1 . \tag{2ii}$$

Condition (2i) states that a new issue may not be ordered while a previously ordered issue is still waiting to be completed. When a capital issue is ordered at time t_i, it takes until $t_i + \Delta$ before new capital can actually be raised. The measurability of s_i with respect to $F_{(t_i+\Delta)-}$ means that the owners may decide on the exact amount of capital to be raised at time $t_i + \Delta$ based on all then available information. They do not need to precommit to any quantity of capital at time t_i when they order the capital issue. Condition (2ii) states that dividends may not be paid during the

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[7] The wiener process is defined on an underlying probability space (Ω, F, P), where we take F to be the standard filtration generated by the Wiener process.

periods between the ordering of a capital issue and the actual capital collection. The condition has important technical merit but also an economic justification. Ruling out simultaneous capital orders and dividend payments is likely to reduce conflicts of incentives between existing and new equity holders. The potential incentive conflicts are not explicitly present in our model and we do not analyze the division of bank value between existing and new equity holders. We simply think of constraint (2ii) as a restriction set by the capital markets.

Bank capital stock as a function of policy $\hat{\pi}$ is denoted $\hat{X}_t^{\hat{\pi}}$, and satisfies the integral dynamics

$$\begin{aligned}\hat{X}_t^{\hat{\pi}} &= \hat{X}_0 + \int_0^t r\hat{X}_{u-}^{\hat{\pi}} du + \int_0^t dY_u - L_t^{\hat{\pi}} + \sum_i s_i^{\hat{\pi}} \mathrm{I}_{\left\{t_i^{\hat{\pi}}+\Delta \le t\right\}} \\ &= \hat{X}_0 + \int_0^t r\hat{X}_{u-}^{\hat{\pi}} du + \int_0^t \mu R_u du + \int_0^t \sigma R_u dW_u - L_t^{\hat{\pi}} + \sum_i s_i^{\hat{\pi}} \mathrm{I}_{\left\{t_i^{\hat{\pi}}+\Delta \le t\right\}}\end{aligned} \qquad (3)$$

where $\mathrm{I}_{\{\cdot\}}$ is the indicator function of the event defined in the parenthesis. This implies that cumulative profits and new issues of capital feed to the capital stock, while dividend payments represent a leakage from the capital stock. Bank capital also earns the risk-free rate[8].

The minimum capital requirement under the current Basel Accord states that bank capital must at all times exceed 8% of the bank's risk weighted assets. We assume that the corrective action from violation of the minimum capital requirement will be liquidation[9]. The model bank therefore only operates up to the liquidation time

$$\hat{\tau}_{\hat{\pi}} = \inf\left\{t : \hat{X}_t^{\hat{\pi}} \le 8\% \cdot R_t\right\}. \qquad (4)$$

The value of bank under policy $\hat{\pi}$ to its owners, given initial level of capital $\hat{X}_0$, is the expected discounted present value of dividends less capital issues until liquidation

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[8] This assumption can be justified in several ways. We could assume that bank capital is explicitly invested in a riskfree asset. Alternatively, we could postulate that any capital the bank has replaces an equivalent amount of borrowing/deposit funding, and that the effective cost of borrowing/deposits to the bank equals the riskfree rate. The latter assumption in turn could be justified by the presence of deposit insurance.

[9] In practice, a violation of the minimum capital requirement will not result in immediate liquidation, but will generate additional costs and constraints to the bank due to increased regulatory surveillance (e.g. Peek and Rosengren, 1997, list the provisions for Prompt Corrective Action specified in the FDICIA). Also the bank's competitive position is likely to be affected. Therefore the bank's owners are likely to lose a significant amount of the bank's economic rent, while our model assumes that all of the economic rent is lost.

$$\hat{V}_{\hat{\pi}}\left(\hat{X}_0\right) = E_{\hat{X}_0}\left[\int_0^{\hat{\tau}_{\hat{\pi}}} e^{-(r+\rho)t} dL_t^{\hat{\pi}} - \sum_i e^{-(r+\rho)\left(t_i^{\hat{\pi}}+\Delta\right)}\left(s_i^{\hat{\pi}} + KR_{t_i^{\hat{\pi}}+\Delta}\right) \mathrm{I}_{\left\{t_i^{\hat{\pi}}+\Delta<\hat{\tau}_{\hat{\pi}}\right\}}\right], \tag{5}$$

where ρ is a positive risk premium on the bank's equity, and K is a non-negative constant representing the cost of capital issuance. (5) implies that the cost from capital issuance is proportional to bank size, as measured by risk weighted assets. The capital control problem is to identify the value of an optimally managed bank

$$\hat{V}(x) = \sup_{\hat{\pi}\in\Pi} \hat{V}_{\hat{\pi}}(x) \tag{6}$$

and an admissible policy which achieves this value. We note that the model has six parameters in total. μ and σ characterize bank returns. r is the risk free rate and ρ is the risk premium associated with bank equity. Δ and K determine the magnitude of the capital market imperfections.

2.2 A normalized model of bank capital ratio

The capital dynamics defined in (3) is not time-homogenous, which makes direct solution of the problem (6) difficult. The problem of capital control can, however, be transformed into a time-homogenous problem of capital ratio control, through a simple normalization. The normalized state variable, the bank capital ratio X, is defined explicitly as

$$X_t = \frac{\hat{X}_t}{R_t} . \tag{7}$$

The following proposition presents the capital ratio control problem and shows its connection to the capital control problem (6).

Proposition 1. *Given a policy $\pi \in \Pi$, let bank capital ratio satisfy*

$$X_t^{\pi} = X_0 + \mu t + \sigma W_t - L_t^{\pi} + \sum_i s_i^{\pi} \mathrm{I}_{\left\{t_i^{\pi}+\Delta\le t\right\}} , \tag{8i}$$

and define the first time of capital ratio violation by

$$\tau_{\pi} = \inf\left\{t : X_t^{\pi} \le 8\%\right\} . \tag{8ii}$$

Define bank equity value, given policy π, by

$$V_{\pi}\left(X_0\right) = E_{X_0}\left[\int_0^{\tau_{\pi}} e^{-\rho t} dL_t^{\pi} - \sum_i e^{-\rho\left(t_i^{\pi}+\Delta\right)}\left(s_i^{\pi} + K\right) \mathrm{I}_{\left\{t_i^{\pi}+\Delta<\tau_{\pi}\right\}}\right], \tag{8iii}$$

where the expectation is conditional on the capital ratio dynamics (8i)*, and let the value function be*

$$V(x) = \sup_{\pi \in \Pi} V_\pi(x)\,. \tag{8iv}$$

Then (6) *can be expressed in terms of* (8iv) *as*

$$\hat{V}\left(\hat{X}_0\right) = R_0 V\left(\frac{\hat{X}_0}{R_0}\right). \tag{9}$$

Moreover, let $\pi^\bullet$ *be the policy which achieves the optimum in* (8iv). *Then the policy which achieves the optimum in* (6), $\hat{\pi}^\bullet$, *can be expressed in terms of* $\pi^\bullet$ *by*

$$L_t^{\hat{\pi}^\bullet} = \int_0^t R_u dL_u^{\pi^\bullet} \tag{10i}$$

$$t_i^{\hat{\pi}^\bullet} = t_i^{\pi^\bullet}, \quad i \geq 1 \tag{10ii}$$

$$s_i^{\hat{\pi}^\bullet} = R_{t_i^{\pi^\bullet}+\Delta} s_i^{\pi^\bullet}, \quad i \geq 1. \tag{10iii}$$

The key to this result is to understand that when π and $\hat{\pi}$ are related through (10), then the capital ratio process X^π given by (8i) is exactly the process $\hat{X}^{\hat{\pi}}/R$, derived from (1) and (3) with the help of Ito's lemma, given that the initial values satisfy $X_0 = \hat{X}_0/R_0$. The complete proof of the proposition is available from the authors upon request.

(9) implies that the objective function of the capital ratio control problem, (8iii), can be interpreted as the value of bank equity *as a percentage of risk weighted assets.* Also, since the capital ratio dynamics in (8i) does not depend on the level of the capital ratio, we may without loss of generality normalize the default point in (8ii) to 0, and interpret X as the excess capital ratio, i.e. capital ratio in excess of 8%. The relation (9) between the solution to the original capital control problem and the solution to the capital ratio control problem is preserved, once we interpret $\hat{X}$ correspondingly as excess capital, i.e. capital stock in excess of 8% of risk weighted assets. We will use this reinterpretation in the following.

2.3 Characterization of optimum

We characterize the value function (8iv) through a set of variational inequalities. For this purpose we define two operators. Let D be the set of real-valued functions on $\mathbf{R}_+$. We define the operator $M{:}D \to D$ by

$$Mf(x) = E_x\left[e^{-\rho\Delta}\sup_s\left[f\left(X_\Delta + s\right) - s - K\right]\mathrm{I}_{\{\tau_0>\Delta\}}\right], \tag{11}$$

where X_Δ is the value at time Δ of X defined by $dX_t = \mu dt + \sigma dW_t$, τ_0 is the first hitting time of 0 of X, and the expectation is conditioned on $X_0 = x$. Operator M which can be interpreted as the expected value of the decision to order new capital immediately, given that the 'continuing value' of the problem is f. Also we define the infinitesimal generator A by

$$Af(x) = \frac{1}{2}\sigma^2 f''(x) + \mu f'(x), \tag{12}$$

for all sufficiently regular f. This may be interpreted as the 'expected instantaneous change in the value of the function f, given no immediate controls are undertaken'.

Now the following characterization of optimum can be established using standard arguments (see e.g. Hojgaard and Taksar, 1999, or Fleming and Soner, 1993).

Proposition 2. *Assume that the value function* (8iv) *satisfies Ito's formula. Then it satisfies the following set of inequalities for all* $x > 0$:

$$V(0) = 0 \tag{13i}$$

$$V(x) \geq MV(x) \tag{13ii}$$

$$(A-\rho)V(x) \leq 0 \tag{13iii}$$

$$V'(x) \geq 1 \tag{13iv}$$

$$\big(V(x) - MV(x)\big)(A-\rho)V(x)\big(V'(x)-1\big) = 0. \tag{13v}$$

(13) is a system of first order conditions to our problem which follow from standard dynamic programming arguments applied to the Bellman equation. With the exception of (13i), they must be understood as functional (in)equalities, i.e. to hold for all positive x (the domain of the state variable in our problem). (13i) follows from 0 capital buffer (remember our reinterpretation of the state variable as excess capital) being an absorbing state in our model. (13ii) holds since the value of immediate order

of new capital can never exceed the value function by definition of the value function. (13iii) holds since applying no control to the capital stock is always an admissible policy. (13iv) must hold since paying dividends is an admissible policy. (13v) states that in an optimum, one of the inequalities must be tight. That is, for all x either taking no action or taking some of the admissible actions must represent the optimal policy.

We note that in Proposition 2, we do not assume that the value function is twice continuously differentiable everywhere. It is well known that the second derivative of a value function in general exhibits a discontinuity at the boundary of the region where impulse control actions are optimal (see Dumas, 1991). This does not prevent Ito's formula from applying, but the differential generator in inequality (13iii) is to be interpreted in terms of left or right derivatives[10].

3 Solutions

Constructing a solution to (13) requires a guess on the form of the solution, i.e. on the order of the 'optimality regions' for each of the policies. Our assumption on the form of the solution in the general case (i.e. under parameter combinations such that capital issues are optimally undertaken) is the following: i) for $x \in (0, u_1]$, it is optimal to immediately order new capital, ii) for $x \in (u_1, u_2)$, it is optimal neither to order new capital nor to pay dividends, and iii) for $x \in [u_2, \infty)$ it is optimal to pay dividends. Furthermore, we expect to have $u_1 < u_2$ and u_2 finite. u_1 may be 0 when capital market imperfections are prohibitively high. Figure 1 illustrates the model with this form of solution.

According to our initial guess, we look for a function that solves (13ii) with equality for $x \leq u_1$, solves (13iii) with equality for $x \in [u_1, u_2]$, and solves (13iv) with equality for $x \geq u_2$. Such function also solves (13i) and (13v). The function is to be continuously differentiable at the impulse control barrier u_1 and twice continuously differentiable at the singular control barrier u_2.

[10] We have presented no proof of sufficiency of the first-order conditions (13). Subject to general restrictions on the form of the solution, such proof of sufficiency can be formulated and can be obtained from the authors upon request.

3.1 General case

We define the following functions:

$$M\left(x;u_2\right)=e^{-\rho\Delta}\left\{\left(x+\mu\Delta+\mu/\rho-K-u_2\right)\Phi\left(\frac{x+\mu\Delta}{\sigma\sqrt{\Delta}}\right)+\sigma\sqrt{\Delta}\varphi\left(\frac{x+\mu\Delta}{\sigma\sqrt{\Delta}}\right)\right.$$
$$\left.-e^{-\frac{2\mu x}{\sigma^2}}\left[\left(-x+\mu\Delta+\mu/\rho-K-u_2\right)\Phi\left(\frac{-x+\mu\Delta}{\sigma\sqrt{\Delta}}\right)+\sigma\sqrt{\Delta}\varphi\left(\frac{-x+\mu\Delta}{\sigma\sqrt{\Delta}}\right)\right]\right\}; \quad (14)$$

$$f_1\left(x;u_2\right)=a_1e^{-d_{1+}(u_2-x)}+a_2e^{-d_{1-}(u_2-x)}, \quad (15)$$

where $a_1=\dfrac{d_{1-}}{d_{1+}d_{1-}-d_{1+}^2}>0$, $a_2=\dfrac{d_{1+}}{d_{1+}d_{1-}-d_{1-}^2}<0$, and $d_{1\pm}=\dfrac{1}{\sigma^2}\left[-\mu\pm\sqrt{\mu^2+2\rho\sigma^2}\right]$;

and

$$f_2\left(x;u_2\right)=\frac{\mu}{\rho}+\left(x-u_2\right). \quad (16)$$

In (14), Φ is the cumulative standard normal distribution and φ is the density of the standard normal distribution. The following result gives the value function (8iv) in terms of these functions, as well as a sufficient condition on the problem parameters for the existence of the general solution.

Proposition 3. *Let* $\dfrac{\partial M\left(x,u_0\right)}{\partial x}\Big|_{x=0}>\dfrac{\partial f_1\left(x,u_0\right)}{\partial x}\Big|_{x=0}$, *where* M *and* f_1 *are given by* (14) *and* (15), *and where*

$$u_0=\frac{2}{d_{1+}-d_{1-}}\log\left(-\frac{d_{1-}}{d_{1+}}\right). \quad (17)$$

Then there exists a solution (u_1,u_2) *to the set of algebraic equations*

$$M\left(u_1,u_2\right)=f_1\left(u_1,u_2\right) \quad (18i)$$

$$\frac{\partial M\left(x,u_2\right)}{\partial x}\Big|_{x=u_1}=\frac{\partial f_1\left(x,u_2\right)}{\partial x}\Big|_{x=u_1} \quad (18ii)$$

satisfying $0<u_1<u_2<u_0$ *and such that* $Mf\left(x,u_2\right)\le f_1\left(x,u_2\right)$ *for all* $0\le x\le u_2$. *In terms of the solution* (u_1,u_2) *to* (18), *the value function* (8iv) *is*

$$V(x) = \begin{cases} M\left(x;u_2\right) & 0 \le x \le u_1 \\ f_1\left(x;u_2\right) & u_1 < x < u_2 \\ f_2\left(x;u_2\right) & u_2 \le x \end{cases} \tag{19}$$

where Mf is given by (14), f_1 *is given by* (15) *and* f_2 *is given by* (16).

The algebraic system (18) is non-linear, but standard numerical optimization procedures (secant method, Newton's method) converge well, given reasonable initial values. We have done the optimization with the Solver™ add-in in Microsoft Excel.

The interpretation of the solution (19) is simple. The value function coincides with the function M in the region where immediate ordering of capital issues is optimal, with the function f_1 in the 'wait and see' region, and with the linear function f_2 in the dividend payment region. Figure 2 illustrates the solution in this general case. Smooth pasting (up to second derivatives) of f_1 and f_2 takes place at u_2, the dividend payment barrier. The solution to (18) in turn imposes smooth pasting (up to first derivatives) of M and f_1 at u_1, the capital issue order barrier. M is more concave at u_1 than is f_1, and the second derivative of the value function at this point experiences a discontinuity.

3.2 Limiting cases

The limiting cases of the model emerge as either the capital issue cost or the capital issue delay approaches zero, or as either of these increases above a critical value so that capital issues are no longer optimal. We find the limiting cases interesting since they help to understand the comparative statics of the general model, and show exactly which of the capital market imperfections drive the qualitative results.

Case I: K or Δ above a critical value

When Δ or K are above their critical values (which depend on other problem parameters) such that capital issuance is no longer optimal, the optimal policy and the value function reduce to those of the Milne and Robertson (1996) model without the capital issue option. This value function is a special case of (19), obtained by setting $u_1 = 0$ and $u_2 = u_0$ as given in (17). The value function takes the form

$$V(x) = \begin{cases} a_1 e^{-d_{1+}(u_0 - x)} + a_2 e^{-d_{1-}(u_0 - x)} & x < u_0 \\ \dfrac{\mu}{\rho} + x - u_0 & x \geq u_0 \end{cases} \tag{20}$$

where a_1, a_2, d_{1+} and d_{1-} are as in (15).

Case II: Δ = 0

In the absence of any capital raising delay, new capital can be issued instantaneously and there is perfect control on the minimum level of capital. Given the opportunity cost of capital, it is clearly optimal to wait until the capital stock falls arbitrarily close to zero before issuing new capital. As zero is an absorbing boundary for the capital stock, however, new issues would have to be implemented before capital stock actually hits zero. Non-surprisingly, an optimal policy in the model without delays does not exist. ε-optimal policies can be constructed which set the capital issue barrier arbitrarily close to 0. One may think of the value function in the $\Delta = 0$ case as the limit of the values associated with such ε-optimal policies.

Taking the limit as Δ approaches 0, the function M in (14) simplifies to

$$M(x; u_2) = x + \mu/\rho - K - u_2,$$

for all $x > 0$. Taking the limit of this as we let the capital issue point x approach zero, we obtain the boundary condition satisfied by the limiting value function in the $\Delta = 0$ case

$$V(0) = \max\left(\lim_{x \to 0+} M(x; u_2)\Big|_{\Delta=0}, 0\right) = \max\left(\mu/\rho - K - u_2, 0\right). \tag{21}$$

This condition now replaces (13i). By the previous reasoning, the capital issue barrier in the limiting case is located at zero, so that the solution only has one free barrier, the dividend barrier. As in the general case, the dividend barrier is also the level up to which the capital ratio is replenished each time a new capital issue is implemented. The solution is given in the following proposition, proven in the Appendix.

Proposition 4. *If $K < \mu/\rho$ - u_0, where u_0 is given by* (17)*, the value function is*

$$V(x) = \begin{cases} a_1 e^{-d_{1+}(\hat{u} - x)} + a_2 e^{-d_{1-}(\hat{u} - x)} & x < \hat{u} \\ \dfrac{\mu}{\rho} + x - \hat{u} & x \geq \hat{u} \end{cases} \tag{22}$$

where $\hat{u} < u_0$ is the unique positive solution for u_2 in the equation

$$a_1 e^{-d_{1+} u_2} + a_2 e^{-d_{1-} u_2} = \frac{\mu}{\rho} - u_2 - K \,. \tag{23}$$

Else, the value function and the barrier $\hat{u}$ *are identical to* (20) *and* (17).

The parametric form of (22) is the same as that of (20), the difference being the location of the barriers u_0 and $\hat{u}$. When the condition of Proposition 4 holds, $\hat{u} < u_0$, and in this case (22) is a left-shifted version of (20).

If both Δ and K are equal to 0, then (23) is solved by $\hat{u} = 0$. This limiting case represents perfect market conditions. In perfect markets, no buffer stocks of capital are held and all profits are immediately paid out as dividends. When losses are realized, the capital to cover the losses is instantaneously raised from capital markets. The controlled capital ratio would be a constant equal to the 8% minimum.

Case III: $K = 0$

We can say little more about this limiting case than about the general case. The limit $K = 0$ does not involve a degeneracy as the limit $\Delta = 0$ does. This is evident from the formula (14) for the function M, where setting K to zero does not influence the qualitative properties of M.

This observation implies that it is the presence of a delay that drives the qualitative nature of the solution, in particular the existence of the non-zero capital issuance barrier. The sole presence of the fixed cost does not generate a positive capital issuance barrier, and will not result in a positive probability of liquidation. We find these observations to support the presence of a capital issue delay, which is precisely the additional ingredient in our modeling strategy, relative to earlier contributions in the banking literature.

4 Comparative statics

In this section we explore the qualitative implications of our model through numerical illustrations.

4.1 Optimal dividend and capital raising policies

The model predicts that optimal dividend policies and equity issuance policies are influenced by the degree of capital market imperfections. In Figure 3 we show the response of the barriers u_2 and u_1 to the capital market imperfections K and Δ. We have drawn the figures with μ, σ and ρ fixed at representative values. We will return to the estimation of these parameters in the following section.

The optimal dividend barrier in the upper picture of Figure 3 is non-decreasing with respect to K and Δ. The optimal choice of the dividend barrier balances the expected cost of new capital issues as well as the expected loss of continuing value from liquidation, against the time value of delayed dividends. The dividend barrier is non-decreasing in the capital issue cost since the latter increases expected capital raising costs. The dividend barrier is non-decreasing in the length of the delay since a longer delay, ceteris paribus, implies a higher probability of liquidation. Also, the dividend barrier is quite sensitive to the introduction of small costs of capital issuance, given that the cost is initially at a low level. When the cost is already sizable, the dividend barrier is relatively insensitive to small increases in that cost.

The optimal capital issue barrier in the lower picture in Figure 3 is non-increasing in the capital issue cost K. This occurs because a higher cost reduces the net benefit from capital issuance. However, the capital issue barrier does not behave monotonically with respect to the delay Δ. When the delay is relatively short, implying a quick access to new capital, the optimal response to an increase in the delay is an increase in the capital raising barrier. In this case a longer delay induces earlier (in time) ordering of new capital. When the delay is relatively long, on the other hand, the optimal response to an increase in the delay is a decrease in the capital issue barrier. This happens because the value of ordering a new capital issue is affected two ways by changes in the delay. First, an increase in the delay increases the probability of liquidation during the delay, ceteris paribus, inducing an increase in the capital issue barrier. Second, the model forbids dividend payments during delay, so that an increase in the delay defers potential dividend payments further into the future, should a capital issue be ordered now, suggesting a decrease in the capital issue barrier. It turns out that for short delays, the former effect dominates, while for sufficiently long delays, the latter effect dominates. Moreover, Figure 3 suggests that the point where the positive response of the capital issue barrier with respect to the delay turns negative is the lower, the higher is the cost from capital issuance. Finally, consistent with Proposition 4, the capital issue barrier converges to zero as the delay approaches zero.

Comparing the dividend and capital issue barriers in Figure 3, we observe that the expected size of a new issue (which is well approximated by the difference between the dividend barrier and the capital issue barrier) increases with the capital raising cost K. The owners optimally issue new equity in larger quantities but less frequently, when the cost of issuance is high, relative to the case where the cost of issuance is low. Such behavior is a natural response to the presence of a capital raising cost, and is descriptive of impulse control policies in general. We will use this observation in the next section to identify plausible values for the parameters Δ and K describing capital market imperfections.

4.2 Value of the capital issue option

The opportunity to issue new equity in our model, being an option, cannot reduce bank value[11]. The value of the capital issue option in the model is the difference between the value functions (19) and (20), for a given initial capital ratio.

The upper picture in Figure 4 shows the value of the capital issue option as a function of the initial capital buffer. We note that the value of the option is monotonically declining in the capital issue cost K, and also in the delay Δ, although this is not shown in the Figure. This is expected since both parameters are pure business constraints. As a function of the initial capital buffer, the value of the capital issue option displays a humped shaped behavior. The option value is at its highest when the capital buffer is somewhat below the optimal capital issue barrier. When capital is close to this barrier, capital issues achieve their intended effectiveness and will be used with high probability (indeed with probability 1 when the capital stock is less than the capital issue barrier). The value of the new issue option decreases as the capital buffer falls significantly below the optimal capital order barrier since it becomes increasingly unlikely that the bank will survive until the end of the delay. As the capital buffer approaches zero, the value of the capital issue option approaches zero as well. The value of the capital issue option also goes down when the capital buffer increases significantly above the capital issue barrier, because here the probability of capital shortages in the near future is low. To conclude, the option to issue new capital, in absolute terms, is most valuable to banks that are

[11] This is not generally true in models with asymmetric information, such as Myers and Majluf (1984).

likely to issue new capital either immediately or in the immediate future, but are still a reasonable distance above their minimum capital requirement.

The lower picture in Figure 4 shows the value of the capital issue option as a percentage of the total (cum-option) value of the bank. The share of bank value attributable to the recapitalization option may be substantial. When the capital issue cost equals 0.25% of risk-weighted assets and capital issuance is subject to a delay of 0.5 years, the value of the capital issue option may be up to 15% of the value of a troubled bank. However, the relative value of the new issue option declines with the capital ratio. The option value is less than 2% of bank value for an otherwise identical bank which is optimally capitalized, i.e. capital buffer equals the dividend barrier (3.4% of risk weighted assets in this example).

The recapitalization option may have value despite substantial capital market imperfections. Figure 4 is based on an annual expected bank income (μ) of 1.0% of the bank's risk weighted assets. The option to issue capital still has value when the cost of a capital issue is 2% of risk weighted assets, i.e. worth of two year's expected profit. This suggests that we should observe bank owners optimally recapitalizing even when this means paying out several years' worth of expected earnings in capital issue related expenses.

5 Calibration and empirical tests

In this section we calibrate the model parameters to data on US banks' accounting returns. We then test the model's ability to explain observed bank capitalizations, both the average level of bank capital and the variation in capital levels across banks.

5.1 Calibration of model parameters

The accounting identity that governs the evolution of bank equity is of the form

$$C_t = C_{t-1} + NI_t - D_t + S_t \,, \tag{24}$$

where C_t is bank equity at time t, NI_t is net income over period (t-1,t), D_t is dividends over period (t-1,t), and S_t is equity issuance over period (t-1,t). Consistent with our model, we decompose net income as

$$NI_t = ROA_t \cdot R_{t-1} + r_t X_{t-1} R_{t-1}, \tag{25}$$

where ROA_t stands for return on (risk weighted) assets over period (t-1,t), and r_t is the riskfree rate over period (t-1,t). R is risk weighted assets and X is bank capital ratio as before. The first summand in (25) is the stochastic return on bank asset portfolio. The second summand is the return on bank equity, where equity is assumed to be invested at the riskfree rate. Also consistent with our modeling assumptions, we impose the condition that bank risk weighted assets grow at the riskfree rate

$$R_t = (1 + r_t) R_{t-1}. \tag{26}$$

Combining (24)-(26), an approximate expression for the discrete dynamics of the bank capital ratio X_t in terms of the accounting variables is

$$\Delta X_t = \frac{C_t}{R_t} - \frac{C_{t-1}}{R_{t-1}} \approx ROA_t - \frac{D_t}{R_{t-1}} + \frac{S_t}{R_{t-1}}, \tag{27}$$

where, according to (25), ROA_t has the expression

$$ROA_t = \frac{NI_t}{R_{t-1}} - r_t X_{t-1}. \tag{28}$$

A comparison of the model capital ratio dynamics (8i) and the discrete dynamics (27) then suggests that we should interpret the model parameters μ and σ in terms of accounting data as

$$\mu = E\left[\frac{NI_t}{R_{t-1}} - r_t X_{t-1}\right],$$

$$\sigma^2 = VAR\left[\frac{NI_t}{R_{t-1}} - r_t X_{t-1}\right].$$

We use annual Compustat data on US commercial banks over the period 1983-2002. We qualify all banks with at least 15 years of data, ending up with a sample at 62 banks. For each bank, we estimate μ and σ from the time series of the bank's ROA, calculated according to (28)[12]. The riskfree rate in this calculation is taken to be the prevailing Fed funds rate. Table 1 summarizes the parameter estimates.

[12] Our estimate for μ is sample mean, and of σ is sample standard deviation. Risk weighted assets, and therefore capital ratios, do not exist in the data prior to 1993. In calculating (28) prior to 1993, we estimate each bank's risk weighted assets based on the average post 1993 risk weighted assets-to-total assets ratio of the bank.

The average estimate of μ across banks is 0.90% and the average estimate of σ is 0.79%. The distribution of μ estimates appears rather symmetric, while the distribution of σ estimates is positively skewed. A related observation is that the μ and σ estimates are mutually negatively correlated (correlation -52%, bottom part of Table 1). Ex ante, a positive correlation between risk and return would be expected. The finding of negative correlation in the data suggests that the sample period is of insufficient length. It can be argued that the sample accommodates only a single credit crisis, that in the early 1990's. As a general pattern, those banks which experienced major credit losses then have low average returns and high volatilities of returns in the data. Many banks which have not experienced a single serious loss episode during the sample period display both a high average return and a low volatility of return. We will return to this later when we interpret our results.

Table 1 also shows the distribution statistics of asset growth rates across banks. The mean growth rate is high, 12.0%. This is significantly higher than the average riskfree rate over the sample period, 5.9%. Our model assumption was that these two rates are of equal magnitude. However, the accuracy of this assumption is hard to measure because a significant amount of the asset growth in the data is likely to have taken place through consolidations with smaller banks. In asset mergers, the dominating bank acquires new assets but also new equity, so that the growth in assets is not a similar burden on capital adequacy as is a greenfield credit expansion. The asset growth rate in our model should more correctly be interpreted as the net growth rate of assets, over the growth rate of equity due to any asset mergers. Hence our model assumption need not be badly mistaken, but we do not possess the data to resolve this issue empirically[13].

The parameter ρ is the bank's equity risk premium. Alternatively, in light of the result $V\left(u_2\right)=\mu/\rho$, ρ may be interpreted as an upper bound on earnings-to-price ratio's among bank's with stationary growth. We have chosen not to estimate ρ for each bank separately, but use a common estimate for ρ equal to 4%. This is

[13] We will show later in Table 3 that bank level asset growth rates are not highly correlated with our model's explanation error, implying that differences in bank level growth rates are not a significant source of model misspecification.

consistent with the equity premium interpretation and implies a maximal price-to-earnings ratio of 25 within our model.

We identify aggregate estimates for Δ and K such that the implied (proportional) capital raising costs as well as the implied frequency of capital issuance are of plausible magnitude. μ and σ we set in this exercise roughly to their average values in the bank level data. In Table 2 we show the results subject to different combinations of Δ and K. Table 2 implies that the delay Δ has only a second-order effect on the proportional cost of capital issues. This is because the delay has only a modest impact on the difference $u_2 - u_1$, which in turn determines the average size of the issue. The difference $u_2 - u_1$ is sensitive to the value of K, and hence we can choose K rather independently from Δ so as to yield a desired proportional cost per capital issue. The empirical evidence on capital issuance costs (see e.g. Lee et al., 1996, or Bajaj et al., 2002) suggests that direct costs of equity issuance amount to up to 10% of issue size. There are likely to be some indirect costs as well, such as the cost of the bank's own effort, so that direct and indirect costs together could amount to over 10% of the proceeds of the issue. Table 2 now implies that a K of roughly 0.25% (of risk weighted assets) achieves proportional capital issuance costs in this range.

Table 2 also shows how Δ influences the expected time between capital issues. Given K fixed at 0.25%, Δ equal to 0.08/0.5/1.0 years implies an expected capital issuance frequency of 18/28/44 years. We do not possess bank level data histories on capital issuance events to evaluate these values empirically. However, 18 years would imply an emergency recapitalization in every second or third recession, which to our intuition is more frequent than most banks are likely to experience. We find 0.5 years a plausible estimate, yielding an average recapitalization frequency of 28 years.

5.2 Level and cross-section of capital ratios

Actual bank capital buffers of the sample banks are summarized in Table 1. The bank level capital buffers are averages over the years 1993-2002. We find it quite reasonable to assume that over this period banks have been fully adjusted to the prevailing minimum capital rules under the Basel Accord of 1988. The average (median) bank capital buffer in the sample, calculated over the 8% minimum, is 5.2% (4.8%). Half of the sample banks have average capital buffers between 3.7% and

6.0%, while no banks have average capital buffer less than 10%. It is noteworthy that 10% capital ratio is also the threshold for the 'Well Capitalized' category introduced by the FDICIA[14].

Figure 5 now plots model dividend barriers (u_2) against actual bank capital buffers for the sample of 62 banks. The average model dividend barrier (2.5%) is substantially lower than the average of actual capital buffers (5.2%). This is reflected in the left picture, where most points are located below the 45 degree line. The model dividend barriers in Figure 5 are based on the calibrated values of Δ and K from the previous section. Yet the averages cannot be reconciled solely through an adjustment of the capital market imperfections. Even in the absence of the capital issue option, the average of model dividend barriers is only 2.9%. Another option to increase model capital ratios were to lower the risk-premium (constant 4% across banks), since this is negatively correlated with model dividend barriers. Lowering the risk-premium to 3% or 2% would generate somewhat higher dividend barriers, but not sufficiently to match the average capital buffer of 5.2% among the sample banks. Also, this would generate implied price-earning ratios that are no longer reasonable.

We do not observe any banks to have a time series capital buffer less than 2%, the value dictated by the FDICIA Well Capitalized category. This raises the issue of whether 10% should be treated as the effective regulatory minimum requirement for banks. The right picture in Figure 5 redefines actual capital buffers as excess of actual capital over 10% of RWA. This yields a scatter plot which is relatively symmetric around the 45 degree line. It is worth noting that this adjustment to capital ratios does not influence the cross-sectional correlation between the model output and actual capital ratios. However, the adjustment is at odds with our model assumptions since violation of the 10% Well Capitalized rule does not imply prompt corrective action or significant restrictions on the bank's activities by its regulators.

The cross sectional correlation between actual capital buffers and model dividend barriers among the sample banks is 39%. Figure 5 reveals that perhaps the most troublesome aspect in the model's performance is the high number of banks with rather low model capital ratios, say around 1%. This is the horizontal formation near

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[14] Peek and Rosengren (1997) contains a description of the Federal Deposit Insurance Corporation Improvement Act from 1991 containing the Well Capitalized category.

the bottom of the pictures. These observations are all due to low return volatility estimates, in the order of 0.5% or less. To illustrate this issue, Figure 6 plots the model dividend barriers against the μ and σ estimates, for the sample of 62 banks. The left picture shows that the model dividend barrier is effectively driven by the asset return volatility, the correlation between the two being 99%. In this picture there is a concentration of firms with volatilities around 0.5%. These firms' model dividend barriers are no higher than 1.5%, and they constitute the horizontal pattern in Figure 5. The right picture in Figure 6 in turn verifies that a high average asset return reduces the model capital buffer, but here the effect is much weaker than the effect of volatility. The correlation in the right picture is -59%.

Table 3 presents correlations between model capital buffers (both the capital issue barrier u_1 and the dividend barrier u_2) and various other variables, including i) the actual capital buffer, ii) the μ and σ estimates, and iii) some possible explanatory factors which are missing from our model. The latter include bank asset size, asset growth rate, and the level of loan loss provisions as a percentage of assets (LLP/assets). All these variables have been calculated as bank level averages over the sample period 1983-2002. Here we observe that asset growth rate and asset size are not materially correlated with model output, while there is a positive correlation between the level of loan loss provisions and the level of model capital buffers. The linkage between these two is the volatility estimate, which is systematically higher with banks that have experienced a high level of credit losses, leading to higher model capital buffers for these banks (correlation between bank level volatility estimates and loan loss provisions is 33%).

Table 3 also shows the correlations between the model residual, i.e. the difference between actual bank capital buffer and the model capital buffer, and the explanatory variables. These can provide signals concerning possible model misspecification. Table 3 shows that asset growth rate is not highly correlated with the model residual, suggesting that our constant growth rate (across banks) assumption does not affect the cross-sectional performance of the model. On the other hand, there appears to be a material negative correlation between bank asset size and the model residual, suggesting that bank size influences bank capitalization through channels other than the mean and volatility of bank returns. The negative correlation means that larger banks hold less capital than small banks, after controlling for possible differences in their Δ and K estimates. In Table 3 we have not varied Δ and K but have used the average estimates discussed previously (moreover, Δ and K do not affect the u_0 estimate)[15]. In the next subsection, we will calculate implied volatilities for large and small banks, taking into account likely differences in capital market imperfections.

Finally, there is a high negative correlation between average loan loss provisions and the model residual. This is a likely consequence of two factors. First, as observed previously, banks with high loan losses tend to have high asset return volatilities (correlation 33%), leading to a positive (negative) correlation between loan loss provisions and model capital buffers (model residuals). Second, credit losses consume bank capital, and loan loss provisions therefore have a direct negative effect on actual capital levels (correlation -31%) and the model residual.

The last observation calls into question whether our basic comparison between bank level time-averages of capital and the model dividend barrier is based on correct variables. In the continuous-time model, the dividend barrier is the highest capital ratio that would ever be observed. Given annual dividend payments, we would expect banks to deviate upwards from that level in-between dividend payments, and even considerable so in years with unexpectedly high profits. On the other hand, banks which have experienced recent losses may have capital ratios anywhere between the barriers u_1 and u_2. It is therefore difficult to define exactly which model quantity should be matched against the time-series averages of bank capital ratios. Our preferred choice is the dividend barrier u_2, but we acknowledge that this may be suspect to error in cases of banks which have experienced serious losses over the sample period[16].

5.3 Implied volatilities for large and small banks

Figure 7 shows the level of volatilities required to replicated the average observed level of capital ratios. For large banks (i.e. those with average assets over 40 bn$),

[15] It appears to us that one possible approach to identifying implied estimates for Δ and K at bank level were to vary these as a function of bank size, so as to generate a zero correlation between the model residual and bank size. There are many degrees of freedom in this exercise, and we will not attempt this at bank level.

[16] u_2 is an upper bound on capital ratios for optimizing banks in our model, and a time-average of capital ratios over time *within the model* would be below u_2. However, there is no single obvious measure of average here because the stationary distribution for the capital ratio in our model is a degenerate one concentrated at 0. This is due to the result that an optimally managed bank in our model bankrupts with probability one in finite time (proof available upon request from the authors).

the average of capital ratios is 3.7% and implies a volatility of roughly 1.0%. For small banks (those with average assets less than 40 bn$), the average capital ratio of 5.7% implies a volatility of roughly 1.25%[17]. These volatilities are around 30% and 50% higher than the average estimated volatilities for large and small banks, 0.76% and 0.80%, respectively. That the adjustment is higher for small banks is consistent with the fact that there is a negative correlation between the model residual and bank size in Table 3. We will discuss the possible rationale for such volatility adjustments in the next section.

5.4 Discussion

The main question that we have set to answer in the empirical part of this paper was: to what extent can actual bank capitalization be explained with an optimizing model of this type, based on empirically plausible parameter values? We have shown that cross-sectional variation in bank accounting return mean and volatility estimates produces a correlation of 39% between model capital ratios and actual capital ratios. In this comparison, the two parameters reflecting capital market imperfections were assumed constant across banks. The ability to vary these parameters across banks will enable a somewhat better model fit. However, the effects of the capital market imperfections are subordinate to the level of volatility, which is the dominant factor driving model capital levels. As for the average level of capital, we have shown that the average volatility adjustment required to replicate the average level of capital buffers is sizable, in the order of 30% for large banks and 50% for small banks, of the estimated level of volatility.

There is considerable uncertainty in the estimates for bank return mean and volatility, which is likely to reduce the model's explanatory power. This uncertainty is reflected in the negative correlations between the μ and σ estimates in Table 1. Also a large number of banks in the sample have quite low asset return volatility estimates. The 25th percentile of the volatility distribution in Table 1 is 0.48%. The median volatility is 0.62% while the mean is 0.79%, so that the volatility estimates are positively skewed. Our Compustat bank return sample is 20 years and

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[17] Our assumption that small banks cannot recapitalize yields a lower bound for the implied volatility estimate. The capital market imperfections for large banks are based on the calibration above.

accommodates only a single major credit crises in the US. The sample is short relative to the estimates of capital issuance frequency in Table 2. All this leads us to believe that historical volatility estimates suffer from a peso problem. According to this hypothesis, many of the low volatility estimates are downward biased. The true volatility distribution would be closer to symmetric and the average volatility would likely be higher. Correcting the volatility bias would potentially improve model performance, both in explaining the average level of bank capital and the variation across banks. This is turn suggests that the generation of accurate forward looking volatility estimates would be crucial to the issue of replicating the level and the cross-section of bank capital ratios.

Model misspecification could explain some of the model error. A potential source of model misspecification, related to the volatility estimation bias, is the nature of bank portfolio return distribution. Our model assumes normally distributed bank returns, while bank portfolio returns, and credit returns in particular, are expected to display a negatively skewed distribution and positive serial correlation. This is evident from bottom-up portfolio models such as CreditMetrics™ (J.P.Morgan, 1997), which simulate bank portfolio returns based on returns on individual counterparties. A positive correlation between counterparty credit standings invariably generates a skewed bank portfolio return distribution. Incorporating non-normality or serial correlation into our model, however, would destroy the analytic tractability currently present in the model. The Compustat bank return data over the period 1983-2002 indicates that there is some negative skewness as well as positive serial correlation in the sample bank returns. A positive volatility adjustment would be required to accommodate these effects within our model. Also, the volatility adjustment would likely be larger for smaller banks to the extent that they have more concentrated portfolios and fewer opportunities to diversify in the marketplace (hence the peso problem is more severe with these banks). This may explain some of the higher implied volatilities for smaller banks in Figure 7.

Another potential bias lies with the calculation of accounting returns used to estimate volatility. Banks in most jurisdictions have some options for income smoothing in the form of loan loss provisions (see e.g. Pain, 2003, for an analysis of banks' provisioning behavior). Discretionary provisions allow banks to distribute credit losses, which are typically realized only during a fraction of quarters over each ‘credit cycle’, more evenly over time[18]. Such behavior will cause bank’s accounting returns to be less volatile compared to banks’ true portfolio (cash) returns. Bank capital requirements, however, apply to an accounting measure of capital, so that it appears entirely justified to base parameter estimates on accounting returns. The issue here is whether a bank optimizes its capital, i.e. its dividend and recapitalization barriers, based on accounting dynamics or the true distribution of its cash flows.

There are several model specification issues that could have significant effects on model output. It is often suggested that market discipline determines bank capitalization. This could be pressure coming from rating agencies, competitors and peers (swap market participants), or customers. Our model abstracts from all market interactions. Whether these omissions matter reduces to the issue of which constituency provides the binding constraint on bank capitalization. Banking theory traditionally holds that customers, to the extent that they enjoy the security of deposit insurance, are not motivated in monitoring banks. Rating agencies’ view is that of debt holders. Jokivuolle and Peura (2004) suggest that regulatory capital buffer (that is what we measure) dominates the economic capital constraint which measures bank riskiness from debtors’ perspective. There is some evidence that competitor reactions, in particularly the access to swap markets, could constitute a binding constraint on bank capital (Jackson et al, 2002).

Banks also face not one but several regulatory compliance requirements. In addition to the Basel Committee’s minimum solvency standard, the FDICIA imposes minimum leverage requirements. We do not model these leverage requirements, which amounts to assuming that risk-based minimum capital requirements are binding over any leverage requirements. This should be the case with banks that have relatively risky portfolios, but need not be the case with banks that have a significant portion of their portfolios invested e.g. in bank assets.

Banks finance themselves by a combination of core deposits and wholesale borrowing, which may come at a different cost. In our empirical implementation, the estimate for μ reflects the average combination in the data. Our model is consistent

[18] Statistical tests which affirm banks’ income smoothing behavior have been provided e.g. by Bhat (1996) and Lobo and Yang (2001).

with the assumption that this combination remains fixed over time. A higher cost of wholesale borrowing is an impediment to credit expansion. Our constant growth rate assumption trivializes this aspect of the model. Yet the low correlation in Table 3 between our model residual and the growth rate suggests that cross-sectional variation in growth rates cannot explain most of the variation in the model residual (moreover, the correlation estimate is negative, which contradicts the explanation that high 'unexplained' capital ratios with some banks were due to their higher than average growth objectives).

Our assumption concerning the penalty from regulatory capital violation is quite extreme. The penalty is of barrier nature, i.e. there is no penalty above the regulatory minimum requirement, but full penalty (loss of entire franchise value) once the requirement is breached. A model where the penalty gradually increases as the capital ratio deteriorates would appear more realistic. The FDICIA e.g. contains 5 categories of capitalization, where a lower category implies more restrictions on bank behavior. These restrictions could be modeled e.g. as additionally costs which would be reflected in the μ parameter (this would then be a gradually increasing function of bank capital ratio). We believe that such adjustment can influence the average level of capital in the model, but is not likely to influence the model's cross-sectional performance which is driven by the volatility parameter.

6 Conclusions

We have presented an optimizing trade-off model of bank capital choice, where the main innovation is to allow for a delay in recapitalization, in addition to a recapitalization cost. This is important in minimum capital analysis since the delay gives rise to a model where banks have recapitalization option but still face liquidation risk. The delay turns out to be a sufficient condition in itself to induce banks to hold positive levels of buffer capital. We have illustrated the option value accruing from the opportunity to issue new capital, and the nature of the optimal capital raising policies.

We have calibrated the model to bank accounting return data, and tested its ability to replicate actual bank capital ratios. This exercise suggests that a dynamic trade-off model of bank capital has the potential to explain much of the cross-sectional differences in bank capitalization by relating these to differences in bank return volatilities (which are ultimately driven by bank portfolio structure). However, the obtained empirical fit is not impressive, and is likely to be seriously weakened by the uncertainty relating to the estimates of bank return volatilities. To a lesser degree, cross-sectional variation in capital ratios can also be explained by differences in capital market imperfections. We have also shown that the implied volatility required to replicate actual bank capital is likely to be higher with small than with large banks, even after acknowledging likely differences in capital market imperfections.

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Appendix: Proofs

Proof of Proposition 3. We attempt to construct a function that solves (13ii) with equality for $x \le u_1$, solves (13iii) with equality for $x \in [u_1, u_2]$, and solves (13iv) with equality for $x \ge u_2$. We also attempt to construct the function so as to be continuously differentiable at the impulse control barrier u_1 and twice continuously differentiable at the singular control barrier u_2.

We first construct the solutions to (13ii)–(13iv), when these hold as equalities. Denoting our candidate solution by f_1, (13iii) becomes

$$\frac{1}{2}\sigma^2 f_1''(x) + \mu f_1'(x) - \rho f_1(x) = 0. \tag{A1}$$

The general solution to this is the exponential function

$$f_1(x) = c_1 e^{d_{1+}x} + c_2 e^{d_{1-}x}, \tag{A2}$$

$$d_{1\pm} = \frac{1}{\sigma^2}\left[-\mu \pm \sqrt{\mu^2 + 2\rho\sigma^2}\right].$$

The general solution to (13iv), denoted f_2, is

$$f_2(x) = x + c \tag{A3}$$

Twice continuous differentiability at u_2 therefore requires that $f_1'(u_2) = 1$ and that $f_1''(u_2) = 0$. Imposing these on (A2) yields

$$f_1(x;u_2) = a_1 e^{-d_{1+}(u_2 - x)} + a_2 e^{-d_{1-}(u_2 - x)}, \tag{A4}$$

$$a_1 = \frac{d_{1-}}{d_{1+}d_{1-} - d_{1+}^2} > 0, \qquad a_2 = \frac{d_{1+}}{d_{1+}d_{1-} - d_{1-}^2} < 0.$$

Also substituting $f_1'(u_2) = 1$ and $f_1''(u_2) = 0$ into (A1) gives

$$f_1(u_2) = \frac{\mu}{\rho}. \tag{A5}$$

Subject to this boundary condition, (A3) becomes

$$f_2(x;u_2) = \frac{\mu}{\rho} + (x - u_2). \tag{A6}$$

Now assume that (13ii) holds with equality, when applied to some concave function f that satisfies $f'(u_2) = 1$. The supremum in (11) is then achieved by $s^* = u_2 - X_\Delta$. Also using (A5), (11) simplifies to

$$M(x;u_2) = e^{-\rho\Delta} E_x\left[\left(X_\Delta + \mu/\rho - K - u_2\right) \mathrm{I}_{\{\tau_0 > \Delta\}}\right]. \tag{A7}$$

Let $\beta = \mu/\rho - K - u_2$, which measures the net benefits from new issues of equity. (A7) further simplifies to

$$\begin{aligned} M(x;u_2) &= e^{-\rho\Delta} E_{x+\beta}\left[X_\Delta \mathrm{I}_{\{\tau_\beta > \Delta\}}\right] \\ &= e^{-\rho\Delta} E_{x+\beta}\left[X_\Delta(\Delta \wedge \tau_\beta) \mathrm{I}_{\{\tau_\beta > \Delta\}}\right], \\ &= e^{-\rho\Delta} \int_\beta^\infty y g(y;\Delta, x+\beta)\, dy \end{aligned} \tag{A8}$$

where $\tau_\beta = \inf\{t : X_t = \beta\}$ and $g(y;\Delta, x+\beta)$ is the density of the absorbed process $X_\Delta(\Delta \wedge \tau_\beta)$ that starts at $x + \beta$. The first equality in (A8) is due to the spatial homogeneity of arithmetic Brownian motion, the second equality follows since the values of X outside the event defined in the indicator function do not affect the expectation, and the third equality is due to the fact that for the absorbed process $X_\Delta(\Delta \wedge \tau_\beta)$, the event in the indicator function is exactly the event that the process has not been absorbed by time Δ. Using the Reflection Principle (see e.g. Borodin and Salminen, 1997), the density can be written as

$$g(y;\Delta, x+\beta) = \varphi\left(y;\mu\Delta + x + \beta, \sigma\sqrt{\Delta}\right) - \exp\left(-\tfrac{2\mu x}{\sigma^2}\right)\varphi\left(y;\mu\Delta - x + \beta, \sigma\sqrt{\Delta}\right),$$

where $\varphi(y;\mu,\sigma)$ denotes the density of a normal distribution with mean μ and variance σ^2, i.e.

$$\varphi(y;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\tfrac{(y-\mu)^2}{2\sigma^2}\right).$$

Substituting this into (A8) and integrating, we get

$$\begin{aligned} M(x;u_2) = e^{-\rho\Delta} &\left\{\left(x + \mu\Delta + \mu/\rho - K - u_2\right)\Phi\left(\frac{x+\mu\Delta}{\sigma\sqrt{\Delta}}\right) + \sigma\sqrt{\Delta}\varphi\left(\frac{x+\mu\Delta}{\sigma\sqrt{\Delta}}\right)\right. \\ &\left. -e^{-\frac{2\mu x}{\sigma^2}}\left[\left(-x + \mu\Delta + \mu/\rho - K - u_2\right)\Phi\left(\frac{-x+\mu\Delta}{\sigma\sqrt{\Delta}}\right) + \sigma\sqrt{\Delta}\varphi\left(\frac{-x+\mu\Delta}{\sigma\sqrt{\Delta}}\right)\right]\right\}, \end{aligned} \tag{A9}$$

where $\Phi(y)$ and $\varphi(y)$ are the cumulative standard normal distribution and its density. Direct substitution shows that $Mf(0) = 0$.

Value matching and smooth pasting conditions at u_1 can now be formulated as

$$M(u_1, u_2) = f_1(u_1, u_2) \tag{A10i}$$

$$\frac{\partial M(x,u_2)}{\partial x}\Big|_{x=u_1} = \frac{\partial f_1(x,u_2)}{\partial x}\Big|_{x=u_1} . \qquad \text{(A10ii)}$$

This non-linear system of equations is quite complicated algebraically, and closed form solutions for u_1 and u_2 do not exist. A sufficient condition for the existence of the solution is provided by Lemma 4. The condition is expressed in terms of a positive barrier u_0 defined by $f_1(0,u_0)=0$, which can be solved for

$$u_0 = \frac{2}{d_{1+} - d_{1-}} \ln\left(-\frac{d_{1-}}{d_{1+}}\right). \qquad \text{(A11)}$$

Lemma 5 then shows that the function (19) constructed in terms of the solution to (A10) is concave and satisfies (13), and hence by a sufficiency theorem coincides with the value function (8iv). This completes the proof of Proposition 3.

Lemma 1 (this will be needed in the proofs of lemmas 2 and 3).

$$E_x\left[X_\Delta \mathrm{I}_{\{\tau_0>\Delta\}}\right] = x + \mu\Delta - \mu\int_0^\Delta p(x,t)\,dt ,$$

where $p(x,t) = P\left[\tau_0 \le t \middle| X_0 = x\right]$.

Proof:

$$\begin{aligned}
E_x\left[X_\Delta \mathrm{I}_{\{\tau_0>\Delta\}}\right] &= E_x\left[X_\Delta \mathrm{I}_{\{\Omega\}}\right] - E_x\left[X_\Delta \mathrm{I}_{\{\tau_0\le\Delta\}}\right] \\
&= E_x\left[X_\Delta\right] - \int_0^\Delta E\left[X_\Delta \middle| X_t = 0\right] \frac{\partial}{\partial t} p(x,t)\,dt \\
&= x + \mu\Delta - \int_0^\Delta \mu(\Delta - t) \frac{\partial}{\partial t} p(x,t)\,dt \\
&= x + \mu\Delta\left(1 - p(x,\Delta)\right) + \mu\int_0^\Delta t \frac{\partial}{\partial t} p(x,t)\,dt \\
&= x + \mu\Delta - \mu\int_0^\Delta p(x,t)\,dt
\end{aligned}$$

The first equality follows from the linearity of the expectation, the second from the law of total probability, the third from the Strong Markov property of arithmetic Brownian motion, the fourth just rearranges, and the fifth follows from integration by parts. End of proof.

Lemma 2. *If* $\beta = \mu/\rho - K - u_2 \geq 0$, *then* $\dfrac{\partial M(x;u_2)}{\partial x} > 0$ *and* $\dfrac{\partial^2 M(x;u_2)}{\partial x^2} < 0$ *for all* $x > 0$.

Proof: (This Lemma is important because it is easy to show that capital issuance can not be optimal unless $\beta \geq 0$.) Let us rewrite the function M starting from (A7) as follows

$$\begin{aligned} M(x;u_2) &= e^{-\rho\Delta} E_x\left[(\beta + X_\Delta) \mathrm{I}_{\{\tau_0 > \Delta\}}\right] \\ &= e^{-\rho\Delta}\left\{\beta\left(1 - p(x,\Delta)\right) + E_x\left[X_\Delta \mathrm{I}_{\{\tau_0 > \Delta\}}\right]\right\} \\ &= e^{-\rho\Delta}\left\{\beta\left(1 - p(x,\Delta)\right) + x + \mu\Delta - \mu\int_0^\Delta p(x,t)\,dt\right\} \end{aligned} \tag{A12}$$

where again $p(x,t) = P\left[\tau_0 \leq t \middle| X_0 = x\right]$ and the third equality utilizes Lemma 1. We know that p satisfies the Kolmogorov backward equation, and that its partial derivatives satisfy

$$\frac{\partial}{\partial x} p(x,t) < 0\,, \quad \frac{\partial^2}{\partial x^2} p(x,t) > 0\,, \quad \frac{\partial}{\partial t} p(x,t) > 0\,,$$

for all $(x,t) \in \mathbf{R}_{++} \times \mathbf{R}_{++}$. We differentiate the final expression in (A12) once and twice with respect to x and obtain

$$\frac{\partial M(x;u_2)}{\partial x} = e^{-\rho\Delta}\left\{-\beta\frac{\partial p(x,\Delta)}{\partial x} + 1 - \mu\int_0^\Delta \frac{\partial p(x,t)}{\partial x}\,dt\right\} > 0$$

$$\frac{\partial^2 M(x;u_2)}{\partial x^2} = e^{-\rho\Delta}\left\{-\beta\frac{\partial^2 p(x,\Delta)}{\partial x^2} - \mu\int_0^\Delta \frac{\partial^2 p(x,t)}{\partial x^2}\,dt\right\} < 0\,,$$

where the inequalities hold for all non-negative β because of the signs of the partials of $p(x,t)$. End of proof.

Lemma 3. *For all* $0 < u_2 \leq u_0$, $M(x;u_2) < f_2(x;u_2) = \dfrac{\mu}{\rho} + x - u_2$.

Proof. Beginning with (A7), we get

$$\begin{aligned}
M(x,u_2) &= e^{-\rho\Delta}E_x\left[\left(X_\Delta + \frac{\mu}{\rho} - K - u_2\right)\mathrm{I}_{\{\tau_0>\Delta\}}\right] \\
&= e^{-\rho\Delta}\left\{E_x\left[X_\Delta \mathrm{I}_{\{\tau_0>\Delta\}}\right] + E_x\left[\left(\frac{\mu}{\rho} - K - u_2\right)\mathrm{I}_{\{\tau_0>\Delta\}}\right]\right\} \\
&= e^{-\rho\Delta}\left\{x + \mu\Delta - \mu\int_0^\Delta p(x,t)\,dt + \left(\frac{\mu}{\rho} - K - u_2\right)\left(1 - p(x,\Delta)\right)\right\} \\
&\le e^{-\rho\Delta}\left\{\frac{\mu}{\rho} + \mu\Delta + x - u_2 - \mu\int_0^\Delta p(x,t)\,dt - p(x,\Delta)\left(\frac{\mu}{\rho} - u_2\right)\right\} \\
&< e^{-\rho\Delta}\left\{\frac{\mu}{\rho} + \mu\Delta + x - u_2\right\} \\
&< \frac{\mu}{\rho} + x - u_2 = f_2(x,u_2)
\end{aligned}$$

where $p(x,t)$ is as defined in Lemma 1. The third equality utilizes Lemma 1, the first inequality is due to setting K to 0, the second inequality is because $u_2 \le u_0 < \mu/\rho$ (this is due to (A5) and (13iv)), and the third inequality is because

$$e^{-\rho\Delta}\left\{\frac{\mu}{\rho} + \mu\Delta\right\} = e^{-\rho\Delta}\left\{\int_0^\infty e^{-\rho t}\mu dt + \mu\Delta\right\} < \int_0^\infty e^{-\rho t}\mu dt = \frac{\mu}{\rho}.$$ End of proof.

Lemma 4. *If* $\frac{\partial M(x,u_0)}{\partial x}\Big|_{x=0} > \frac{\partial f_1(x,u_0)}{\partial x}\Big|_{x=0}$, *then there exists a solution* (u_1,u_2) *to* (A10) *satisfying* $0 < u_1 < u_2 < u_0$ *such that* $M(x,u_2) \le f_1(x,u_2)$ *for all* $0 \le x \le u_2$.

Proof. Suppose that the condition $\frac{\partial M(x,u_0)}{\partial x}\Big|_{x=0} > \frac{\partial f_1(x,u_0)}{\partial x}\Big|_{x=0}$ holds.

i) In (A11) we have defined u_0 such that $f_1(0,u_0) = 0$. Therefore $M(0,u_0) = 0 = f_1(0,u_0)$. The condition implies that there is a positive x within $(0,u_0)$ such that $M(x,u_0) > f_1(x,u_0)$. On the other hand, by Lemma 3, $M(u_0,u_0) < f_1(u_0,u_0) = \frac{\mu}{\rho}$. This implies that $M(x,u_0)$ must cross $f_1(x,u_0)$ from above within the interval $0 < x < u_0$.

ii) From (A4) we get that $\frac{\partial f_1(x,u_2)}{\partial u_2} < 0$, so that $M(0,u_2) = 0 < f_1(0,u_2)$ for $0 < u_2 < u_0$. By Lemma 3, we also have $M(u_2,u_2) < f_1(u_2,u_2) = \frac{\mu}{\rho}$ for $0 < u_2 \le u_0$. From (A9) and (A4) we get (given positive Δ)

$$\lim_{u_2 \to 0} M\left(u_2, u_2\right) = 0 < \frac{\mu}{\rho} = \lim_{u_2 \to 0} f_1\left(0, u_2\right).$$

A necessary condition for the general solution is that $\mu/\rho - K - u_2 \geq 0$. Then by Lemma 2, $M(x,u_2)$ is increasing and concave in x, while $f_1(x,u_2)$ given by (A4) is also increasing in x. Combined with the previous inequality, these imply that one can always find a positive u_2 in the interval $(0,u_0)$ such that $M\left(x,u_2\right) < f_1\left(x,u_2\right)$ for $0 \leq x \leq u_2$.

iii) Following from i) and ii), by the continuity of $Mf(x,u_2)$ and $f_1(x,u_2)$ with respect to x and u_2, there will exist a u_2 in the interval $(0,u_0)$ such that $M\left(u_1,u_2\right) = f_1\left(u_1,u_2\right)$ for some $0 < u_1 < u_2$, while $M\left(x,u_2\right) \leq f_1\left(x,u_2\right)$ for all $0 \leq x \leq u_2$. But at this choice of (u_1,u_2), continuous differentiability of $M(x,u_2)$ and $f_1(x,u_2)$ with respect to x implies that $\frac{\partial Mf\left(x,u_2\right)}{\partial x}\Big|_{x=u_1} = \frac{\partial f_1\left(x,u_2\right)}{\partial x}\Big|_{x=u_1}$. This is because two continuously differentiable functions which coincide in the interior of their domain, but do not cross, must posses equal derivatives at the point of where the functions coincide. End of proof.

Lemma 5. *Assume that a solution to* (A10) *as described in Lemma* 4 *exists and that V is defined by* (19). *Then V is a concave solution to* (13) *and satisfies Ito's formula.*

Proof: V is concave: By construction $V''(x) = 0$ for $x \geq u_2$. Differentiating f_1 given by (15) three times shows that $\frac{\partial^3 f_1\left(x;u_2\right)}{\partial x^3} > 0$ on $x \in (u_1,u_2)$. Therefore f_1 has an increasing second derivative on (u_1,u_2), which combined with the fact that $\frac{\partial^2 f_1\left(x;u_2\right)}{\partial x^2}\Big|_{x=u_2} = 0$ implies that f_1 and therefore V is concave on (u_1,u_2). We also know from Lemma 2 that $M(x;u_2)$ is globally concave w.r.t. x, so that V is concave on $(0,u_1)$. Equality of first derivatives of M and f_1 at u_1 then implies that V is globally concave.

V satisfies Ito's formula because each of the component solutions is twice continuously differentiable, and V satisfies the smooth pasting conditions at the barriers u_1 and u_2.

V solves (13):

(13i): $V(0) = 0$ because $M(0;u_2) = 0$.

(13ii): $V(x) = M(x;u_2) = MV(x)$ for $0 \le x \le u_1$ by construction. By Lemma 4, $V(x) = f_1(x;u_2) \ge M(x;u_2) = MV(x)$ for $u_1 \le x \le u_2$, and by Lemma 3, $V(x) = f_2(x;u_2) > M(x;u_2) = MV(x)$ for $x \ge u_2$.

(13iii): For $0 < x < u_1$, we have $V(x) = M(x;u_2)$, and we get from Itô's formula

$$e^{-\rho\tau} M(X_\tau;u_2) = M(x;u_2) + \int_0^\tau (A-\rho) M(X_s;u_2)\,ds + \int_0^\tau \frac{\partial M(X_s;u_2)}{\partial X_s} \sigma dW_s \ ,$$

where τ is a stopping time defined by $\tau = \tau_\varepsilon \wedge \varepsilon$, for some $\varepsilon > 0$ such that $x - \varepsilon > 0$, $x + \varepsilon < u_1$, and $\tau_\varepsilon = \inf\{t \ge 0 : X_t \notin (x-\varepsilon, x+\varepsilon) | X_0 = x\}$. Taking expectations and noting that the last term is a martingale because $M(x;u_2)$ is concave, we obtain

$$E_x\left[e^{-\rho\tau} M(X_\tau;u_2)\right] = M(x;u_2) + E_x\left[\int_0^\tau (A-\rho) M(X_s;u_2)\,ds\right],$$

where $E_x\left[e^{-\rho\tau} M(X_\tau;u_2)\right]$ is the value of waiting until τ prior to ordering a new capital issue. Because immediate ordering of capital is the optimal action at $x < u_1$, we have

$$M(x;u_2) \ge E_x\left[e^{-\rho\tau} M(X_\tau;u_2)\right].$$

Combining the last two equations and the fact that $V(x) = M(x;u_2)$ for $x < u_1$, we get

$$E_x\left[\int_0^\tau (A-\rho) V(X_s;u_2)\,ds\right] \le 0 \,.$$

A limit operation then gives us

$$\lim_{\varepsilon \to 0} \frac{1}{E_x[\tau]} E_x\left[\int_0^\tau (A-\rho) V(X_s;u_2)\,ds\right] = (A-\rho)V(x) \le 0 \,,$$

for $x < u_1$.

For $u_1 < x < u_2$, we have $(A\text{-}\rho)\,V(x) = 0$ by construction.

For $x > u_2$, we have $V(x) = \mu/\rho + x - u_2$, $V'(x) = 1$, $V''(x) = 0$, so that

$$(A-\rho)V(x) = \frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \rho V(x) = \mu - \rho\left(\frac{\mu}{\rho} + x - u_2\right) = u_2 - x < 0$$

for all $x > u_2$.

(13iv): $V'(x) = 1$ for $x \ge u_2$ by construction. That $V'(x) > 1$ for $x < u_2$ follows from the concavity of V (proved above).

(13v): Follows directly from construction.

End of proof.

Proof of Proposition 4. The first-order condition is as in the general case, but with (13i) and (13ii) replaced by (21). (13iii) and (13iv) are solved as in the general case, and smooth pasting at u_2 is enforced as previously. This yields formulas (A4) and (A6). The barrier u_2 is solved by substituting (A4) into the boundary condition (21). Assuming that the maximum in (21) is achieved by the first term, u_2 is determined from the equation

$$g(u_2) \equiv a_1 e^{-d_{1+}u_2} + a_2 e^{-d_{1-}u_2} = \frac{\mu}{\rho} - K - u_2 . \tag{A13}$$

$g{:}R_+ \to R$ defined in (A13) has the following properties: i) $g(0) = \mu/\rho$; ii) $g'(u) < 0$; iii) $g(u) \to -\infty$ as $u \to \infty$; iv) $g''(0) = 0$; v) $g'''(u) < 0$. The convergence in iii) is exponential, and g is concave by iv) and v). It follows that when K is positive, we have $g(0) = \frac{\mu}{\rho} > \frac{\mu}{\rho} - K$, and because the right-hand side of (A13) is linear in u_2, (A13) is solved by a unique $\hat{u} > 0$.

For the maximum in (21) to be achieved by the first term, the solution to (A13) must satisfy

$$g(\hat{u}) = \frac{\mu}{\rho} - K - \hat{u} > 0 , \tag{A14}$$

i.e. $g(u_2)$ must be positive at the point of intersection with the function $\mu/\rho - K - u_2$. Setting $g(u) = 0$, we get that $u = u_0$, where u_0 is given by (17). Then a necessary and sufficient condition for (A14) to hold is that $\mu/\rho - K - u_0 > 0$, or equivalently that $K < \mu/\rho - u_0$, which is the condition in the proposition. In this case $\hat{u} < u_0$. End of proof.

Table 1. Distribution of capital buffers, bank return parameters, and asset growth rates

The sample is 62 commercial banks with at least 15 years of annual Compustat data over the period 1983-2002. Capital buffers are bank level averages over 1993-2002, while ROA's and asset growth rates are bank level averages over 1983-2002.

Distributions	capital ratio	average ROA (μ)	st.dev. ROA (σ)	asset growth
Minimum	2.14 %	-0.33 %	0.22 %	-1.75 %
25th percentile	3.70 %	0.72 %	0.48 %	6.92 %
Median	4.83 %	0.90 %	0.62 %	10.72 %
Average	5.19 %	0.90 %	0.79 %	12.05 %
75th percentile	6.04 %	1.11 %	1.06 %	15.33 %
Max	15.84 %	1.88 %	2.72 %	41.29 %
St.dev.	2.16 %	0.35 %	0.47 %	7.60 %
Correlations	**capital ratio**	**average ROA (μ)**	**st.dev. ROA (σ)**	**asset growth**
capital ratio	100 %	-19 %	44 %	-13 %
average ROA (μ)		100 %	-52 %	21 %
volatility ROA (σ)			100 %	-1 %
asset growth				100 %

Table 2. Expected cost and frequency of capital issues

$E[s_i]$ is the expected net (of costs) size of a capital issue. $K/(K+E[s_i])$ is the expected proportional cost of a capital issue, i.e. total cost divided by gross proceeds. $E(t_i-t_{i-1})$ is the expected time interval between successive capital issues (the expectation is conditioned on the assumption that the bank will not be liquidated during the first issue delay). Fixed parameter values: $\mu = 1.0\%$, $\sigma = 1.0\%$, $\rho = 4.0\%$.

Δ (years)	K	u_1	u_2	$E[s_i]$	$K/(K+E[s_i])$	$E(t_i-t_{i-1})$ (years)
0.08	0.10 %	0.9 %	2.4 %	1.5 %	6 %	8
0.08	0.25 %	0.8 %	2.7 %	1.8 %	12 %	18
0.08	0.50 %	0.7 %	2.9 %	2.2 %	19 %	36
0.08	1.00 %	0.6 %	3.1 %	2.5 %	29 %	68
0.5	0.10 %	1.6 %	3.2 %	1.6 %	6 %	11
0.5	0.25 %	1.3 %	3.4 %	2.0 %	11 %	28
0.5	0.50 %	1.1 %	3.5 %	2.4 %	17 %	57
0.5	1.00 %	0.9 %	3.7 %	2.8 %	27 %	122
1	0.10 %	1.7 %	3.5 %	1.8 %	5 %	16
1	0.25 %	1.4 %	3.7 %	2.3 %	10 %	44
1	0.50 %	1.1 %	3.7 %	2.7 %	16 %	103
1	1.00 %	0.6 %	3.8 %	3.2 %	24 %	291

Table 3. Model capital buffers and correlations with explanatory factors

Averages and correlations based on the 62 banks in the sample.

	capital issue barrier (u_1)	dividends barrier (u_2)	dividends barrier (u_0)	difference (actual buffer-u_2)	difference (actual buffer-u_0)
Average	0.85 %	2.52 %	2.88 %	2.67 %	2.31 %
Correlation with					
actual capital buffer	39 %	39 %	41 %	65 %	47 %
average ROA (μ)	-59 %	-59 %	-59 %	31 %	41 %
volatility ROA (σ)	98 %	99 %	99 %	-39 %	-58 %
asset growth	-2 %	-2 %	-3 %	-11 %	-8 %
asset size	6 %	6 %	5 %	-38 %	-34 %
LLP/assets	37 %	38 %	36 %	-62 %	-62 %

Figure 1. Illustration of model structure

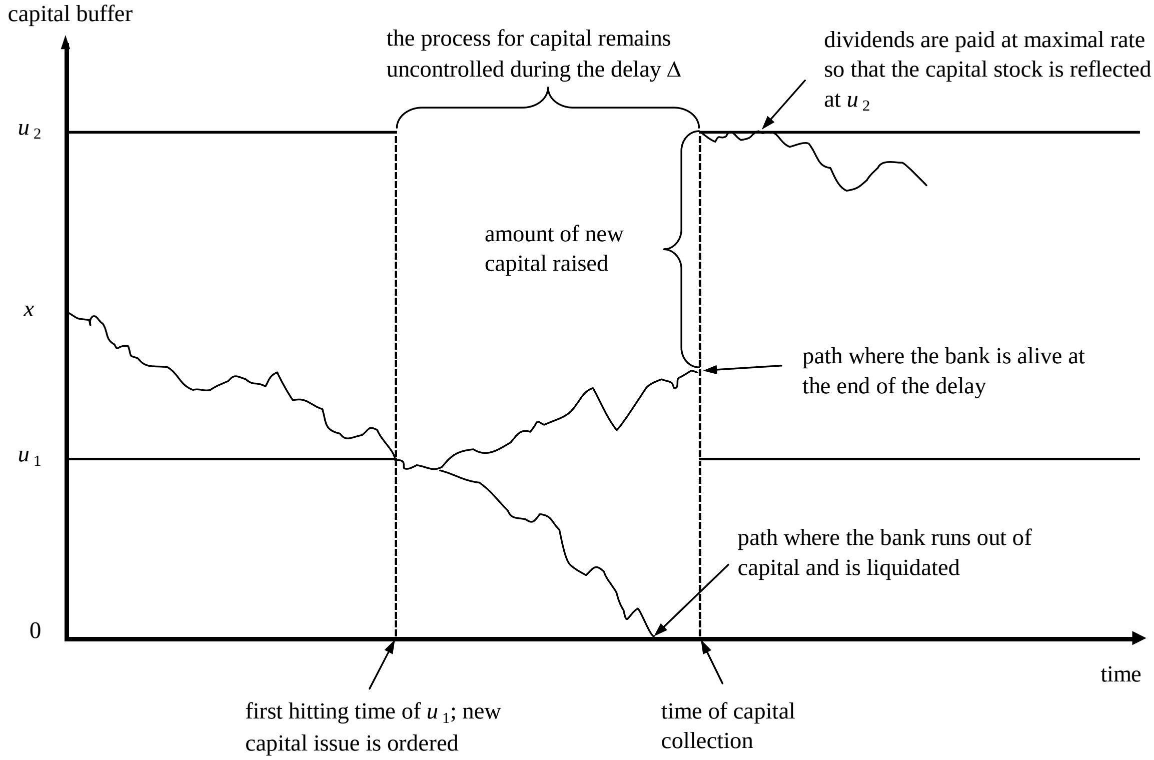


Figure 2. Construction of the value function

The figure shows the components of the value function (19). Parameter values $\mu =$ 1.0%, $\sigma = 1.0\%$, $\rho = 4.0\%$, $\Delta = 0.5$ years, $K = 1.0\%$. The optimal capital issuance barrier u_1 is 0.90% and the optimal dividend barrier u_2 is 3.66%. These are marked with vertical dotted lines. The optimal dividend barrier in the absence of the equity issue option u_0 is 3.80%.

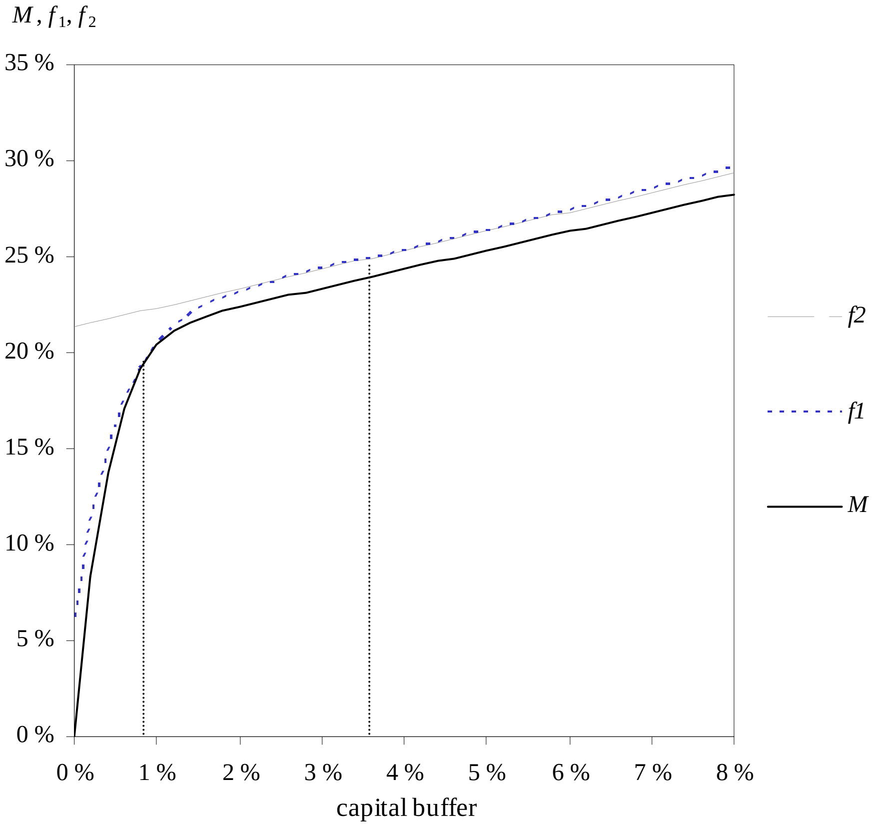
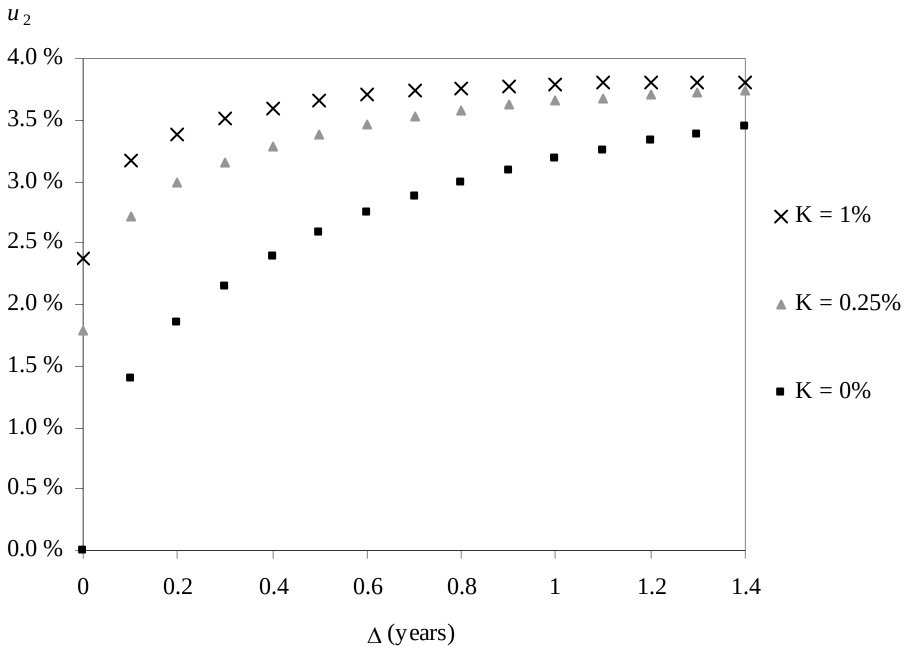


Figure 3. Optimal capital issue and dividend barriers

The upper picture shows the optimal dividend barrier u_2 as a function of the capital market imperfections Δ and K. The lower picture shows the optimal capital issue barrier u_1. Fixed parameter values $\mu = 1.0\%$, $\sigma = 1.0\%$, $\rho = 4.0\%$.



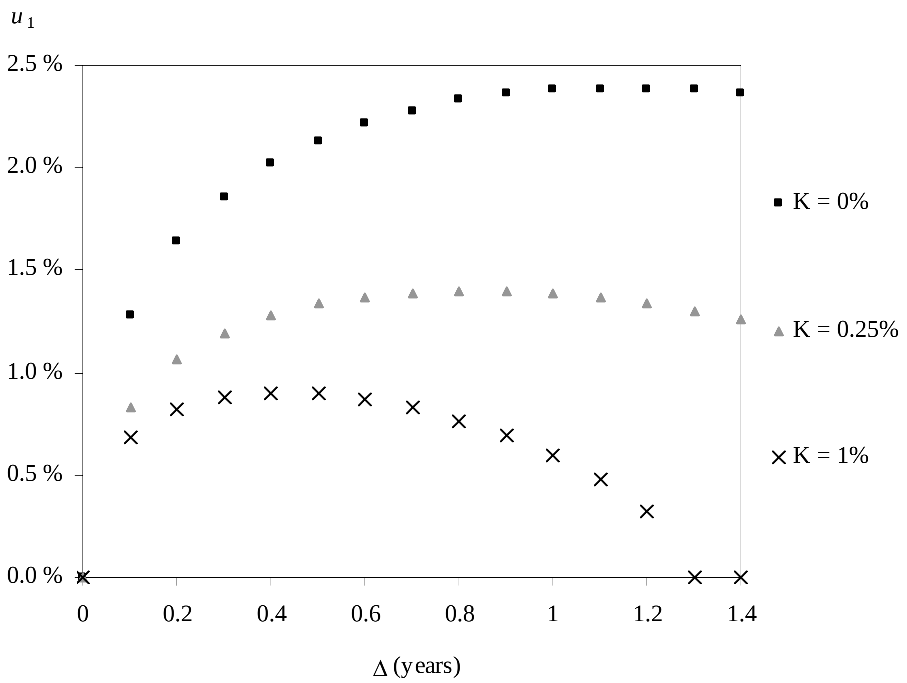
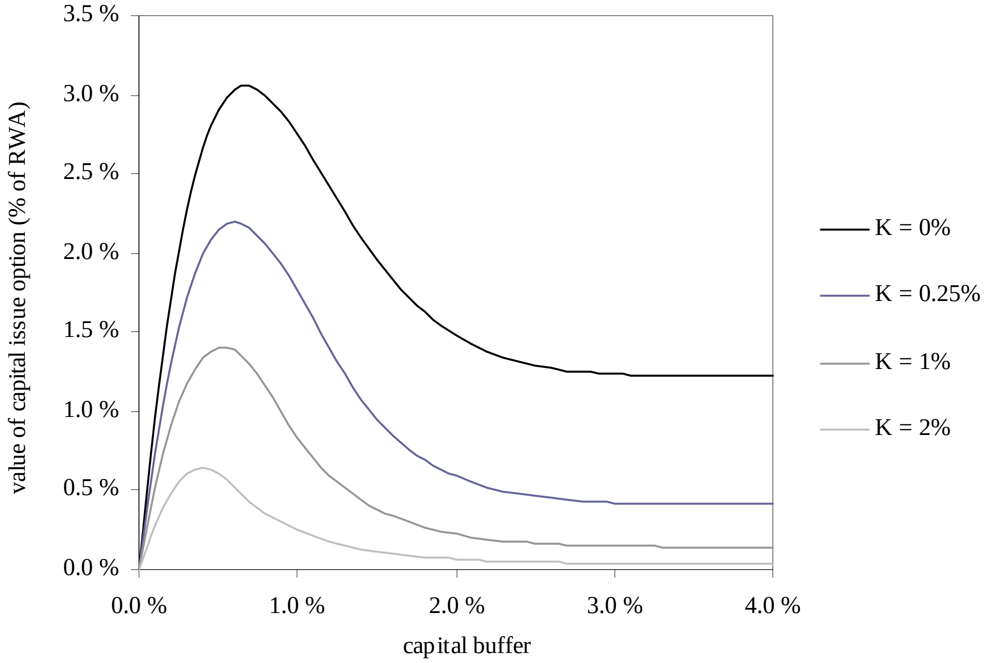


Figure 4. Value of the capital issue option

The upper picture shows the value of the capital issue option as a percentage of risk-weighted assets. The lower picture shows the value of the capital issue option as a percentage of bank value. Fixed parameter values: $\mu = 1.0\%$, $\sigma = 1.0\%$, $\rho = 4.0\%$, $\Delta = 0.5$ years.



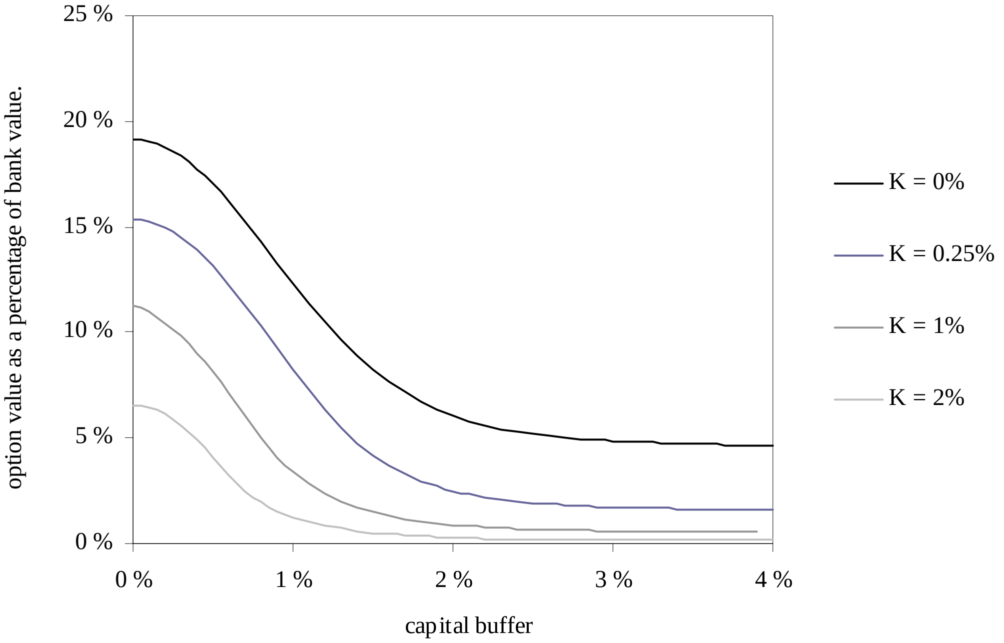


Figure 5. Actual vs. model capital ratios for the sample banks

The left picture shows model capital buffer (u_2) plotted against actual capital buffer, when the floor over which the buffer is calculated is the Basel regulatory minimum, 8%. In the right picture, the floor is 10%, which corresponds to the well-capitalized bank rule in the US. The correlation between actual and model capital buffers is 39% in both cases.

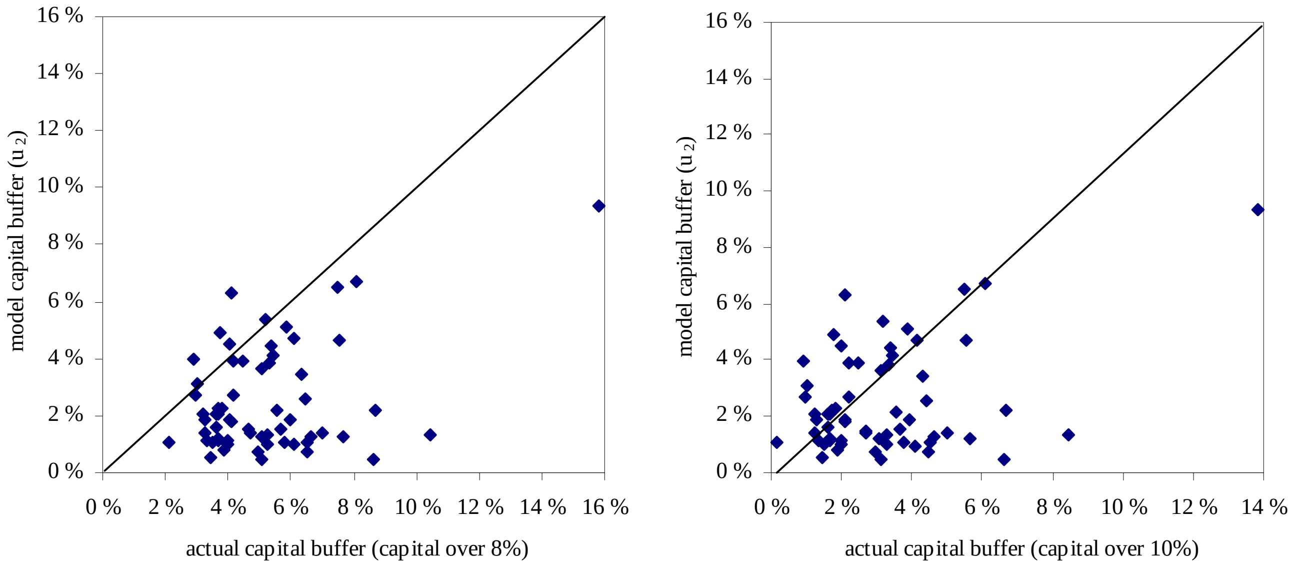


Figure 6. Drivers of the model capital buffer

The left picture shows model capital buffer (u_2) plotted against the σ estimate, for the sample of 62 banks. The right picture shows model capital buffer (u_2) plotted against the μ estimate. The correlation in the left (right) picture is 99% (-59%).

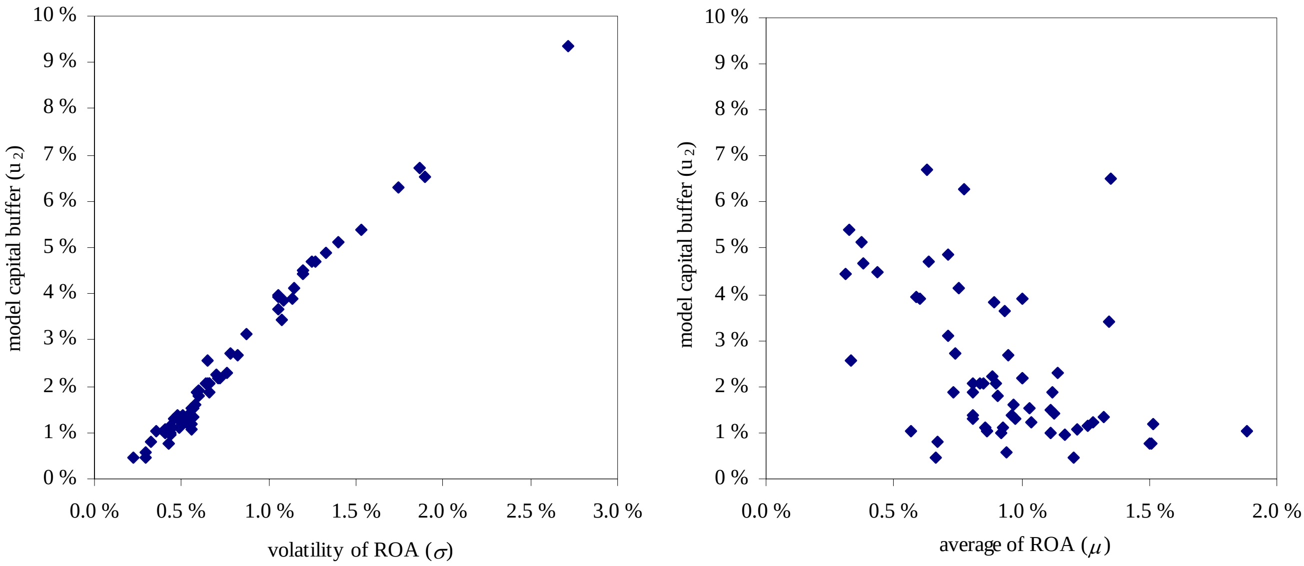


Figure 7. Implied volatilities for large and small banks

The dividend barrier (u_2) plotted against the volatility estimate (σ) for large banks ($\Delta = 0.5$, $K = 0.25\%$) and small banks (capital market imperfections are prohibitively high). The dotted lines indicate the implied volatilities corresponding to the actual capital ratios of large and small banks, 5.71% and 3.69%, respectively.